35. Integration by substitution

35.1. Introduction

The chain rule provides a method for replacing a complicated integral by a simpler integral. The method is called integration by substitution ("integration" is the act of finding an integral). We illustrate with an example:

35.1.1 Example Find $\int \cos(x+1)\,dx$.

Solution We know a rule that comes close to working here, namely, $\int \cos x\,dx = \sin x + C$, but we have $x+1$ instead of just x. If we let $u = x+1$, then

$$du = \frac{du}{dx}\,dx = (1)dx = dx$$

(see 26), so

$$\int \cos(x+1)\,dx = \int \cos u\,du = \sin u + C = \sin(x+1) + C,$$

where in the middle we have used the known rule (with the letter u replacing the letter x). □

In the solution, we substituted the simple u for the (slightly) more complicated $x+1$ and this resulted in an integral that we knew how to find.

35.1.2 Example Find $\int \cos(2x+3)\,dx$.

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Solution As in the first example, the rule $\int \cos x\, dx = \sin x + C$ comes close to working.

Let $u = 2x + 3$, so that $du = \dfrac{du}{dx}\, dx = 2dx$.

Then, inserting 1 in the form $\frac{1}{2} \cdot 2$ and moving the $\frac{1}{2}$ to the outside, we get

$$\begin{aligned}\int \cos(2x+3)\, dx &= \int \cos(2x+3)\left(\tfrac{1}{2}\cdot 2\right) dx \\ &= \tfrac{1}{2}\int \underbrace{\cos(2x+3)}_{\cos u}\, \underbrace{2dx}_{du} \\ &= \tfrac{1}{2}\int \cos u\, du \\ &= \tfrac{1}{2}\sin u + C \\ &= \tfrac{1}{2}\sin(2x+3) + C\end{aligned}$$

□

35.2. Theorem

In the last example (35.1.2), let $f(x) = \cos x$ and $g(x) = 2x+3$. Then $f(g(x)) = \cos(2x+3)$, $g(x) = u$ and $g'(x) = 2$. The critical step in the solution was the use of the equality

$$\int \cos(2x+3)\, 2dx = \int \cos u\, du,$$

which, in terms of f and g, is

$$\int f(g(x))\, g'(x) dx = \int f(u)\, du.$$

Since

$$g'(x) dx = \frac{du}{dx}\, dx = du$$

this last integral equation appears to be valid. However, there is reason to be suspicious. Earlier, we decided to write $\int f(x)\, dx$ to stand for the most general antiderivative of f. The dx is just part of the notation; we have given no justification for treating it as though it were an actual differential as we are doing here. The following consequence of the chain rule provides the justification:

> INTEGRATION BY SUBSTITUTION. If f and g are functions, then
>
> $$\int f(g(x))\, g'(x) dx = \int f(u)\, du,$$
>
> where $u = g(x)$.

Verification: The equality amounts to saying that $\int f(u)\, du$ is the most general antiderivative of $f(g(x))g'(x)$. This can be verified by showing that

$$\frac{d}{dx}\left[\int f(u)\, du\right] = f(g(x))g'(x).$$

Directly from the definition of the integral of f, we have

$$\frac{d}{dx}\left[\int f(x)\, dx\right] = f(x).$$

So using this rule together with the chain rule, we get

$$\frac{d}{dx}\left[\int f(u)\,du\right] = f(u)\frac{du}{dx} = f(g(x))g'(x),$$

as desired.

35.3. Strategy

For integration by substitution to work, one needs to make an appropriate choice for the u substitution:

> STRATEGY FOR CHOOSING U. Identify a composition of functions in the integrand. If a rule is known for integrating the outside function, then let u equal the inside function.

In Example 35.1.2, the expression $\cos(2x+3)$ is the composition of $2x+3$ (first function applied) and $\cos x$ (second function applied). Since we had a rule for integrating the outside function $\cos x$, we chose to let u equal the inside function $2x+3$.

35.3.1 Example Find $\displaystyle\int x^2\left(x^3+5\right)^9\,dx$.

Solution We see the expression $\left(x^3+5\right)^9$, which is the composition of x^3+5 (inside function) and x^9 (outside function). We have the power rule for integrating the outside function, so we let u be the inside function:

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Let $u = x^3 + 5$, so that $du = 3x^2 dx$.

(We have stopped writing the intermediate step $du = (du/dx)dx$.) The goal is to rewrite the given integral as an integral involving only u's (no x's). If we move the x^2 over next to the dx, then this is almost du; we are lacking only a factor of 3. We arrange for this 3 by multiplying by 1 in the form $\frac{1}{3} \cdot 3$ and moving $\frac{1}{3}$ to the outside:

$$\begin{aligned}
\int x^2 \left(x^3+5\right)^9 \, dx &= \int \left(x^3+5\right)^9 \, x^2 dx \\
&= \int \left(x^3+5\right)^9 \left(\tfrac{1}{3} \cdot 3\right) x^2 dx \\
&= \tfrac{1}{3} \int \underbrace{\left(x^3+5\right)^9}_{u^9} \underbrace{3x^2 dx}_{du} \\
&= \tfrac{1}{3} \int u^9 \, du \\
&= \tfrac{1}{3} \left(\frac{u^{10}}{10}\right) + C \\
&= \frac{(x^3+5)^{10}}{30} + C.
\end{aligned}$$

□

35.4. Examples

35.4.1 Example Find $\displaystyle\int \frac{x}{x^2+1} \, dx$.

Solution An appropriate composition is easier to see if we rewrite the integrand:

$$\int \frac{x}{x^2+1}\,dx = \int \left(x^2+1\right)^{-1} x\,dx.$$

The expression $\left(x^2+1\right)^{-1}$ is the composition of x^2+1 (inside function) and x^{-1} (outside function).

Let $u = x^2+1$, so that $du = 2x\,dx$.

We lack the factor of 2 needed to make up the du, so we mentally insert 1 in the form $\frac{1}{2}\cdot 2$ and move the $\frac{1}{2}$ to the outside:

$$\begin{aligned}\int \frac{x}{x^2+1}\,dx &= \tfrac{1}{2}\int \underbrace{\left(x^2+1\right)^{-1}}_{u^{-1}}\underbrace{2x dx}_{du}\\ &= \tfrac{1}{2}\int u^{-1}\,du\\ &= \tfrac{1}{2}\ln|u| + C\\ &= \tfrac{1}{2}\ln\left(x^2+1\right) + C.\end{aligned}$$

(The absolute value sign was omitted in the final answer since x^2+1 is always positive.) □

35.4.2 Example Find $\int xe^{x^2}\,dx$.

Solution The expression e^{x^2} is the composition of x^2 (inside function) and e^x (outside function).

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Let $u = x^2$, so that $du = 2x\,dx$.

We have

$$\begin{aligned}\int xe^{x^2}\,dx &= \tfrac{1}{2}\int \underbrace{e^{x^2}}_{e^u}\underbrace{2xdx}_{du} \\ &= \tfrac{1}{2}\int e^u\,du \\ &= \tfrac{1}{2}e^u + C \\ &= \tfrac{1}{2}e^{x^2} + C.\end{aligned}$$

□

An integral that nobody can evaluate. If we modify the preceding example slightly by omitting the factor x we get

$$\int e^{x^2}\,dx.$$

Let's try the substitution that we used before: $u = x^2$, $du = 2x\,dx$. Multiplying by 1 in an appropriate form in order to try to arrange for the du, we get

$$\int e^{x^2}\,dx = \int e^{x^2}\left(\tfrac{1}{2x}\cdot 2x\right)dx.$$

The problem here is that we cannot move $\frac{1}{2x}$ to the outside (only constants slip outside the integral sign). This substitution will not work.

One can suppose that there might be some other way to find this integral. However, it has been shown that this integral cannot be expressed using elementary functions. In other

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words, e^{x^2} has no antiderivative that can be expressed by using trigonometric, inverse trigonometric, exponential, or logarithmic functions in combination with $+$, $-$, $\times$, $\div$, and $\sqrt[n]{\cdot}$.

This shows how much harder integration is, in general, than differentiation. We have rules of differentiation that can be used to find the derivative of *any* function built up of elementary functions. Yet here is an example of a very simple function that has no elementary antiderivative (and there are plenty of others, too).

We return now to examples of integrals that we *can* find.

35.4.3 Example Find $\int (\sin t) \sec^2(\cos t)\, dt$.

Solution The composition $\sec^2(\cos t)$ has outside function $\sec^2 t$, which we know how to integrate, so we let u be the inside function $\cos t$:

Let $u = \cos t$, so that $du = -\sin t dt$.

We have

$$\begin{aligned}\int (\sin t) \sec^2(\cos t)\, dt &= -\int \sec^2(\cos t)\, (-\sin t) dt \\ &= -\int \sec^2 u\, du \\ &= -\tan u + C \\ &= -\tan(\cos t) + C.\end{aligned}$$

□

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35.4.4 Example Find $\displaystyle\int \frac{3\cos(\pi/x)}{x^2}\,dx$.

Solution The composition $\cos(\pi/x)$ has outside function $\cos x$, which we know how to integrate, so we let u be the inside function π/x:

$$\text{Let } u = \frac{\pi}{x}, \text{ so that } du = -\frac{\pi}{x^2}\,dx.$$

We associate the $1/x^2$ with the dx to start forming the du, and then finish the process by multiplying by $-\pi$ on the inside and by its reciprocal $-1/\pi$ on the outside. Also, the 3 is not required, so it is moved to the outside:

$$\begin{aligned}
\int \frac{3\cos(\pi/x)}{x^2}\,dx &= -\frac{3}{\pi}\int \cos(\pi/x)\left(-\frac{\pi}{x^2}\right)dx \\
&= -\frac{3}{\pi}\int \cos u\,du \\
&= -\frac{3}{\pi}\sin u + C \\
&= -\frac{3}{\pi}\sin(\pi/x) + C.
\end{aligned}$$

□

35.4.5 Example Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

Solution A suitable composition is difficult to see here because the outside function is too simple, but $\ln x$ is a composition with outside function x and inside function $\ln x$:

Let $u = \ln x$, so that $du = \dfrac{1}{x}\,dx$.

We have

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \int \ln x \left(\frac{1}{x}\right) dx \\ &= \int u\,du \\ &= \frac{u^2}{2} + C \\ &= \frac{(\ln x)^2}{2} + C.\end{aligned}$$

□

35.4.6 Example Find $\int x^5 \sqrt[3]{x^3+1}\,dx$.

Solution The expression $\sqrt[3]{x^3+1}$ is the composition of x^3+1 (inside function) and $\sqrt[3]{x}$ (outside function).

Let $u = x^3 + 1$, so that $du = 3x^2\,dx$.

Since we need a factor of x^2 to help make up the du, we break x^5 up into x^3x^2 and associate x^2 with dx. We need to change everything into u's (no x's), so we use the substitution to

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write the leftover factor x^3 as $u-1$.

$$\begin{aligned}
\int x^5 \sqrt[3]{x^3+1}\, dx &= \int \left(x^3+1\right)^{1/3} x^3\, x^2 dx \\
&= \tfrac{1}{3} \int \underbrace{\left(x^3+1\right)^{1/3}}_{u^{1/3}} \underbrace{x^3}_{u-1} \underbrace{3x^2 dx}_{du} \\
&= \tfrac{1}{3} \int u^{1/3}(u-1)\, du \\
&= \tfrac{1}{3} \int \left(u^{4/3} - u^{1/3}\right)\, du \\
&= \tfrac{1}{3} \left(\frac{u^{7/3}}{7/3} - \frac{u^{4/3}}{4/3}\right) + C \\
&= \frac{\left(x^3+1\right)^{7/3}}{7} - \frac{\left(x^3+1\right)^{4/3}}{4} + C.
\end{aligned}$$

□

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35 – Exercises

35 – 1 Find $\int \cos(8x-3)\,dx$.

35 – 2 Find $\int x^4 e^{x^5+3}\,dx$.

35 – 3 Find $\int 4x^2\sqrt{1-x^3}\,dx$.

35 – 4 Find $\int \tan\theta\,d\theta$.

HINT: Write $\tan\theta$ in terms of sine and cosine.

35 – 5 Find $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$.

35 – 6 Find $\int \frac{3x+2}{x^2+1}\,dx$

HINT: First rewrite the integrand as a sum of fractions.